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REGULAR DIFFERENTIAL GAME

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We discuss the interconnection between different methods of constructing decision strategies in encounter-evasion differential games within the framework of that formalization of these games which was the topic of the survey papers [1, 2].

1. We consider a dynamic system described by the differential equation

$$x^{\cdot} = f(t, x, u, v), \quad u \in P, \quad v \in Q \tag{1.1}$$

Here x is the system's phase vector, u and v are the control vectors of the first and the second players, P and Q are bounded and closed sets. The function $f(t, x, u, v)$ is continuous, continuously differentiable in x, and satisfies the condition

$$\|f\| \leqslant \lambda(1 + \|x\|), \quad \lambda = \text{const} \tag{1.2}$$

where $\|x\|$ is the Euclidean norm of vector x.

By the game's hypotheses, in a certain auxiliary space $\{t, m\}$, where m is a parameter, we are given a closed set M whose sections by the hyperplanes $t = \text{const}$ will be denoted by $M(t)$. We shall assume that the sets $M(t)$ are bounded. Let $T(t_0)$ be the set of those values of $t \geqslant t_0$ for which the sections $M(t)$ are nonempty. We are given a function $\rho(t, x, m)$, bounded and continuous for all possible values of x and of $m \in M(t)$ $(t \in T(t_0))$ and continuously differentiable in x for

$$\alpha < \rho(t, x, m) < \beta \tag{1.3}$$

Let

$$\omega(t, x) = \min_{m \in M(t)} \rho(t, x, m) \tag{1.4}$$

The game's outcome is determined by the functional

$$\varphi(x[\cdot]) = \inf_{t \in T(t_0)} \omega(t, x[t]) \tag{1.5}$$

which is minimized by the first player and maximized by the second. (A dot in the place of the argument of any function signifies that we are not dealing with the value of the function for any value of the argument but are dealing with this function as a whole as an element of a functional space). In particular, if the space $\{m\}$ coincides with space $\{x\}$ and

$$\rho(t, x, m) = \|x - m\| \tag{1.6}$$

and so we have $\alpha = 0$ in (1.3), then we obtain the standard encounter-evasion game [1] with set M in the space $\{t, x\}$. In this case we can desist from the condition of boundedness of the sets $M(t)$.

In order not to fall outside the framework of pure strategies [1] $U \div u(t, x)$ and

$V \div v(t, x)$ (*) we shall assume that in all positions $\{t, x\}$ of interest to us the saddle-point condition of the small game [1]

$$\min_{u \in P} \max_{v \in Q} s'f(t, x, u, v) = \max_{v \in Q} \min_{u \in P} s'f(t, x, u, v) \tag{1.7}$$

is fulfilled for every choice of vector s. (Unless stipulated, the vectors under consideration are treated as column-vectors; the prime denotes transposition).

Otherwise it is necessary to transform the constructions used below in accordance with the rules from (1, 2] for analogous games examined in the classes of pairs: mixed strategy-mixed strategy or strategy-counterstrategy.

Thus, by formalizing the motions $x[t, t_0, x_0, U]$ and $x[t, t_0, x_0, V]$ as the limits of the Euler polygonal lines [1] $x_\Delta[t, t_0, x_*, U, v[\cdot]]$ and $x_\Delta[t, t_0, x_*, V, u[\cdot]]$, being the solutions of the equations

$$x_\Delta^{\cdot} = f(t, x_\Delta, u(\tau_i, x_\Delta[\tau_i]), v[t]) \quad (\tau_i \leqslant t < \tau_{i+1})$$

$$x_\Delta^{\cdot} = f(t, x_\Delta, u[t], (x_\Delta^{\cdot}, x_\Delta[\tau_i^*])) \quad (\tau_i^* \leqslant t < \tau_{i+1}^*)$$

we obtain the following two problems for the first and second players, respectively.

Problem 1. Given the position $\{t_0, x_0\}$. Find the optimal minimax strategy $U^\circ \div u^\circ(t, x)$ which satisfies the condition

$$\sup_{x[\cdot]} \varphi(x[\cdot, t_0, x_0, U^\circ]) = \min_{U} \sup_{x[\cdot]} \varphi(x[\cdot, t_0, x_0, U])$$

Problem 2. Given the position $\{t_0, x_0\}$. Find the optimal maximin strategy $V^\circ \div v^\circ(t, x)$ which satisfies the condition

$$\inf_{x[\cdot]} \varphi(x[\cdot, t_0, x_0, V^\circ]) = \max_{V} \inf_{x[\cdot]} \varphi(x[\cdot, t_0, x_0, V])$$

2. The game being analyzed is characterized by the following theorem on the alternative, which transforms the alternative formulated in [1] for the standard encounter-evasion game. Let $T(t_0, \vartheta)$ be the collection of all values of t for the interval $[t_0, \vartheta]$ for which the sections $M(t)$ are nonempty. We denote

$$\varphi_\vartheta(x[\cdot]) = \min_{t \in T(t_0, \vartheta)} \omega(t, x[t]) \tag{2.1}$$

Then the following statement is valid under condition (1.7).

Theorem 2.1. Whatever be the position $\{t_0, x_0\}$ and the numbers $\vartheta > t_0$ and c, either there exists a strategy $U_c \div u_c(t, x)$ which guarantees the inequality

$$\varphi_\vartheta(x[\cdot, t_0, x_0, U_c]) \leqslant c \tag{2.2}$$

or there exists a strategy $V \div v(t, x)$ which guarantees the inequality

$$\varphi_\vartheta(x[\cdot, t_0, x_0, V]) > c \tag{2.3}$$

Stated differently: either there exists a strategy $V_c \div v_c(t, x)$ which guarantees the inequality

$$\varphi_\vartheta(x[\cdot, t_0, x_0, V_c]) \geqslant c \tag{2.4}$$

or there exists a strategy $U \div u(t, x)$ which guarantees the inequality

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*) Editor's Note. The symbol $\div$ denotes the correspondence between the strategy and the function prescribing this strategy.

$$\varphi_\vartheta (x[\cdot,\ t_0,\ x_0,\ U]) < c \tag{2.5}$$

From this theorem follows the existence of a saddle point for the differential game set together from Problems 1 and 2 by choosing $\varphi = \varphi_\vartheta$ in them.

3. Suppose that for some choice of $\vartheta > t_0$ and c there exists a strategy U_c guaranteeing the fulfillment of condition (2.2). Then, according to [1], there exists a u-stable bridge $W_u^{\vartheta,c}$ which forms a closed set in the space $\{t, x\}$, passes through the position $\{t_0, x_0\}$, and terminates on the set $L_c^* (\vartheta)$ at the instant ϑ where

$$L_c^* (t) = [\{t, x\} : \omega (t, x) \leqslant c]$$

i.e. the section $W_u^{\vartheta,c} (\vartheta)$ lies in $L_c^* (\vartheta)$. (Without loss of generality we assume that the set $L_c^* (\vartheta)$ is nonempty because otherwise we can replace ϑ without anything being essentially changed, by the largest of the numbers $\vartheta^* < \vartheta$ which satisfy the condition). The strategy $U_c \div u_c (t, x)$, extremal [1] to the bridge $W_u^{\vartheta,c}$, retains every motion $x [t, t_0, x_0, U_c]$ on $W_u^{\vartheta,c}$ until this motion meets with $L_c^* (\tau)$ for $\tau \leqslant \vartheta$, which achieves the guaranteed fulfillment of condition (2.2).

Conversely, suppose that for some choice of $\vartheta > t_0$ and c there exists a strategy V_c guaranteeing the fulfillment of condition (2.4). Then there exists a v-stable bridge $W_v^{\vartheta,c}$ which forms a closed set in the strip $t_0 \leqslant t \leqslant \vartheta$ of space $\{t, x\}$, passes through the position $\{t_0, x_0\}$ and does not intersect the set

$$L_c = [\{t, x\} : t \in T (t_0, \vartheta),\ \omega (t, x) < c]$$

The strategy $V_c \div v_c (t, x)$, extremal to bridge $W_v^{\vartheta,c}$, retains every motion $x [t, t_0, x_0, V_c]$ on $W_v^{\vartheta,c}$ until the instant ϑ, which achieves the guaranteed fulfillment of condition (2.4).

The assertion on the existence of the needed bridge $W_u^{\vartheta,c}$ or $W_v^{\vartheta,c}$ bears the nature of a pure existence theorem. However, we can seek a bridge $W_u^{\vartheta,c}$ or $W_v^{\vartheta,c}$ on the basis of one of the well-known, more or less effective approaches. In particular, the bridge $W_u^{\vartheta,c}$ or $W_v^{\vartheta,c}$ can be sought by the method of extremal aiming [2] or in the form of an *a priori* stable bridge [2]. A comparison of these two approaches in their application to the encounter-evasion game being considered here is the subject of the present paper. Generally speaking, the methods mentioned for constructing the bridges $W_u^{\vartheta,c}$ and $W_v^{\vartheta,c}$ lead to different results. However, here we shall consider the cases, regular for each of these methods, when both methods lead to like results and each of the construction methods furnishes optimal strategies U° and V°.

4. Paper [2] describes the construction of bridges $W_u^{\vartheta,c}$ and $W_v^{\vartheta,c}$ on the basis of auxiliary program constructions of extremal aiming, corresponding to a position encounter-evasion game considered in the framework of mixed strategies. We describe a version of these constructions [3], corresponding to the position encounter-evasion game being considered within the framework of pure strategies U and V.

Suppose that we have chosen a function $\nu_t (dv)$, weakly-measurable in t $(t_* \leqslant t \leqslant \vartheta)$, whose values $\nu (dv)$ are probability measures on Q. A program $\{\eta_t \mid \nu_t\}_\Pi$ is defined as the set of all possible functions $\eta_t (du, dv)$ weakly-measurable in t $(t_* \leqslant t \leqslant \vartheta)$, whose values $\eta (du, dv)$ are probability measures on $P \times Q$ satisfying the conditions

$$\int_P \eta_t (du, dv) = \nu_t (dv) \tag{4.1}$$

for almost all t. The program motions $x(t, t_*, x_*, \eta_.)$ are defined as the solutions of the differential equation

$$x^{\cdot}=\iint\limits_{P\,Q} f(t, x, u, v)\,\eta_t(du, dv), \quad x(t_*)=x_* \tag{4.2}$$

An auxiliary program problem is formulated as follows.

Problem 4.1. The initial position $\{t_*, x_*\}$ is given and the number $\vartheta > t_*$ has been chosen. We are required to find τ_0, a maximizing program $\{\eta_t \mid \nu_t^{\circ}\}_{\Pi}$, and an optimal control $\eta_t^{\circ} \in \{\eta_t \mid \nu_t^{\circ}\}_{\Pi}$, satisfying the condition

$$\min_{t\in T(t_*,\vartheta)}\ \max_{\{\eta_t\mid\nu_t\}_{\pi}}\ \min_{\eta_t} \omega(t, x(t, t_*, x_*, \eta_.)) = \omega(\tau_0, x(\tau_0, t_*, x_*, \eta_.^{\circ})) = \varepsilon_0(t_*, x_*) \tag{4.3}$$

We assume the fulfillment of the following condition.

Condition 4.1. With a selected value of ϑ, for every initial position $\{t_*, x_*\}$ in a region $\varepsilon_0(t_*, x_*) \in (\alpha, \beta)$, $t_* \leqslant \vartheta$, we can find at least one minimizing instant τ_0 from (4.3) such that every maximizing program from (4.3), corresponding to this τ_0, contains only a single, essentially optimal control $\eta_t^{\bullet}$, and the corresponding minimizing value m_0 from (1.4), satisfying, therefore, the condition

$$\omega(\tau_0, x(\tau_0, t_*, x_*, \eta_.^{\circ})) = \rho(\tau_0, x(\tau_0, x(\tau_0, t_*, x_*, \eta_.^{\circ}), m_0)) = \min_{m\in M(\tau_0)} \rho(\tau_0, x(\tau_0, t_*, x_*, \eta_.^{\circ}), m)$$

is also unique.

The set of values of τ_0 satisfying this condition is denoted by $T^{\circ}(t_*, \vartheta)$. When Condition 4.1 is fulfilled, the optimal control for Problem 4.1 with $\varepsilon_0(t_*, x_*) \in (\alpha, \beta)$ and $\tau_0 \in T^{\circ}(t_*, \vartheta)$ satisfies the following maximin condition which in the given maximin case of Problem 4.1 corresponds to the maximum principle [4] for the usual program problems of optimal control. The equality

$$s'(t) \iint\limits_{PQ} f(t, x(t, t_*, x_*, \eta_.^{\bullet}), u, v)\,\eta_t^{\circ}(du, dv) = \max_{v\in Q}\ \min_{u\in P} s'(t) f(t, x(t, t_*, x_*, \eta_.^{\circ}), u, v)$$

is valid for almost all $t \in [t_*, \vartheta]$. Here $s(t)$ is the solution of the problem

$$s^{\cdot} = -\left[\iint\limits_{PQ} \left\{\frac{\partial f}{\partial x}\right\}' \eta_t^{\bullet}(du, dv)\right] s \tag{4.4}$$

$$s(\tau_0) = \left[\frac{\partial\omega}{\partial x}\right]_{(\tau_0)} = \left[\frac{\partial\rho}{\partial x}\right]_{\{\tau_0,\, x(\tau_0, t_*, x_*, \eta_.^{\bullet}),\, m_0\}}$$

where $\{\partial f / \partial x\}$ is the Jacobi matrix computed on the optimal minimizing motion $x(t, t_*, x_*, \eta_.^{\circ})$.

In the case of a properly linear equation of motion (1.1), when

$$x^{\cdot} = A(t)\,x + f(t, u, v) \tag{4.5}$$

the requirements of Condition 4.1 can be weakened, by requiring only that for every optimal motion $x(t, t_*, x_*, \eta_.^{\circ})$ from one and the same program the values of $s(t)$

from (4.4) turn out to be one and the same every time. If in the case of Eq. (4.5) we have further that the sets $M(t)$ are convex, then such a weakened Condition 4.1 is necessarily fulfilled automatically. Assuming Condition 4.1 as fulfilled, we say that the original position encounter-evasion game set together from Problems 1 and 2 when $\varphi = \varphi_\vartheta$ is regular (I) if the following further condition is fulfilled.

We introduce the following notation. The symbol $S(t_*, x_*; \tau_0)$ denotes the set of all vectors $s = s(t_*)$ from (4.4) corresponding to all possible optimal solutions η_t° of Problem 4.1 for a given position $\{t_*, x_*\}$ in region $\varepsilon_0(t_*, x_*) \in (\alpha, \beta)$ and with the noted minimizing value τ_0. Further, let $\{t_*, x_*\}$ once again be some position in the region $\varepsilon_0(t_*, x_*) \in (\alpha, \beta)$ and let $\{t^*, x^*\}$ $(t^* > t_*)$ be a certain position close to it. Let τ_0^* be any minimizing instant from Problem 4.1, corresponding to the position $\{t^*, x^*\}$; let $\{\eta_t \mid \nu_t^*; \tau_0^*\}_\Pi$ be a maximizing program from a condition of form (4.3), corresponding to the position $\{t_*, x_*\}$ but for the instant τ_0^*, and let $\eta_t^* \in \{\eta_t \mid \nu_t^*; \tau_0^*\}_\Pi$ be the minimizing control from a condition of form (4, 3) for the instant τ_0^* but for the position $\{t^*, x^*\}$; finally, let m^* be the minimizing value in the condition

$$\omega(\tau_0^*, x(\tau_0^*, t^*, x^*, \eta_.^*)) = \rho(\tau_0^*, x(\tau_0^*, t^*, x^*, \eta_.^*), m^*) = \rho(\tau_0^*, x(\tau_0^*, t^*, x^*, \eta_.^*), m)$$

for min $m \in M(\tau_0^*)$. The symbol $s(t^*)$ denotes a vector of form (4.4) wherein we should only replace τ_0 by τ_0^*, $x(\tau_0, t_*, x_*, \eta_.^\circ)$ by $x(\tau_0^*, t^*, x^*, \eta_.^*)$ and m° by m^*. The symbol $S^*(t_*, x_*)$ denotes the set of all vectors s which are the limits for vectors of form $s(t^*)$ under all possible choices of positions $\{t^*, x^*\} \to \{t_*, x_*\}$ and minimizing instants τ_0^* corresponding to these positions. In particular, if the sets $M(t)$ are continuous in $t \in [t_0, \vartheta]$ in the Hausdorff metric, then as the set $S^*(t_*, x_*)$ we should choose simply the set of all vectors $s = s(t_*)$ (4.4) corresponding to all possible optimal solutions η_t° of Problem 4.1 for the position given.

Condition 4.2. The following two requirements must be fulfilled whatever be the position $\{t_*, x_*\}$ $(t_* \leqslant \vartheta)$ from the region $\varepsilon_0(t_*, x_*) \in (\alpha, \beta)$.

1°. For every choice of $v^* \in Q$ we can find $\tau_0 \in T^\circ(t_*, \vartheta)$ and a vector f^*

$$f^* \in \mathrm{co}\,\{f(t_*, x_*, u, v^*),\ u \in P\}$$

satisfying the inequalities

$$s'f^* \leqslant \max_{v \in Q} \min_{u \in P} s'f(t_*, x_*, u, v) \tag{4.6}$$

for all values of $s \in S(t_*, x_*; \tau^\circ)$.

2°. For every choice of $u^* \in P$ we can find a vector

$$f^* \in \mathrm{co}\,\{f(t_*, x_*, u^*, v),\ v \in Q\}$$

satisfying the inequalities

$$s'f^* \geqslant \max_{v \in Q} \min_{u \in P} s'f(t_*, x_*, u, v) \tag{4.7}$$

for all values of $s \in S^*(t_*, x_*)$.

If the game is regular, the sets

$$W_u^{\vartheta, c} = [\{t, x\} : t_0 \leqslant t \leqslant \vartheta,\ \varepsilon_0(t, x) \leqslant c]$$

with $c \in [\alpha, \beta)$ are u-stable bridges, while the sets

$$W_v^{\vartheta, c} = [\{t, x\} : t_0 \leqslant t \leqslant \vartheta,\ \varepsilon_0(t, x) \geqslant c]$$

with $c \in (\alpha, \beta]$ are v-stable bridges. Therefore, the strategies U_c and V_c, extremal to these bridges $W_u^{\vartheta,c}$ and $W_v^{\vartheta,c}$, ensure, respectively, inequalities (2.2) with $c \in [\alpha, \beta)$ and inequalities (2.4) with $c \in (\alpha, \beta]$; when $c \in (\alpha, \beta)$ they form, therefore, a saddle point $\{U_c, V_c\}$ of the game being considered for the initial position $\{t_0, x_0\}$ for which $\varepsilon_0(t_0, x_0) \in (\alpha, \beta)$.

5. Another way to construct the bridges $W_u^{\vartheta,c}$ and $W_v^{\vartheta,c}$ is to seek them in the form of a priori stable bridges [2] on the basis of suitable integral manifolds. Let us start with that construction which corresponds to the notion of upper program from [5]. Let $\mu(du \mid t, x, v)$ $(v \in Q)$ be some function whose values $\mu(du)$ are probability measures on P. We construct the Euler polygonal lines $x_\Delta^*[t, t_0, x_*, \mu(\cdot)]$ as solutions of the differential equation

$$x_\Delta^{\cdot *} = \int_P f(t, x_\Delta^*, u, v[\tau_i])\,\mu(du \mid \tau_i, x_\Delta^*[\tau_i], v[\tau_i]) \qquad (5.1)$$
$$(\tau_i \leqslant t < \tau_{i+1})$$

and next we set up the integral manifold X_u^* from all possible limits $x^*(t, t_0, x_0, \mu(\cdot))$ $(t_0 \leqslant t \leqslant \vartheta)$ for the convergent sequences of such Euler polygonal lines (under the conditions $\lim \sup_i (\tau_{i+1} - \tau_i) = 0$). We can verify that the set

$$W_u^\vartheta = [\{t, x\} : t_0 \leqslant t \leqslant \vartheta,\ x = x^*(t, t_0, x_0, \mu(\cdot)) \in X_u^*] \qquad (5.2)$$

forms a u-stable bridge. Analogously, by choosing a function $\nu(dv \mid t, x, u)$ $(u \in P)$ whose values $\nu(dv)$ are probability measures on Q, we construct the integral manifold X_v^* from all possible limits $x^*(t, t_0, x_0, \nu(\cdot))$ $(t_0 \leqslant t \leqslant \vartheta)$ for the Euler polygonal lines $x_\Delta^*[t, t_0, x_*, \nu(\cdot)]$ which are solutions of the differential equation

$$x_\Delta^{\cdot *} = \int_Q f(t, x_\Delta^*, u[\tau_i], v)\,\nu(dv)\,\tau_i, x_\Delta^*[\tau_i], u[\tau_i]) \qquad (5.3)$$

The set

$$W_v^\vartheta = [\{t, x\} : t_0 \leqslant t \leqslant \vartheta,\ x = x^*(t, t_0, x_0, \nu(\cdot)) \in X_v^*] \qquad (5.4)$$

forms a v-stable bridge. The auxiliary problems connected with the bridges W_u^ϑ from (5.2) and W_v^ϑ from (5.4) are formulated as follows.

Problem 5.1. An initial position $\{t_0, x_0\}$ is given and the number $\vartheta > t_0$ chosen. We are required to find the function $\mu^0(du \mid t, x, v)$ and the value of τ_0 satisfying the condition

$$\min_{\mu(\cdot)} \min_{t \in (t_0, \vartheta)} \max_{x(\cdot)} \omega(t, x(t, t_0, x_0, \mu(\cdot))) = \omega(\tau_0, x(\tau_0, t_0, x_0, \mu^\circ(\cdot))) = \varepsilon^*(t_0, x_0) \qquad (5.5)$$

Problem 5.2. An initial position $\{t_0, x_0\}$ is given and the number $\vartheta > t_0$ chosen. We are required to find the function $\nu^0(dv \mid t, x, u)$ satisfying the condition

$$\max_{\nu(\cdot)} \min_{x(\cdot)} \min_{t \in T(t_0, \vartheta)} \omega(t, x(t, t_0, x_0, \nu(\cdot))) = \omega(\tau_0, x(\tau_0, t_0, x_0, \nu^\circ(\cdot))) = \varepsilon_*(t_0, x_0) \qquad (5.6)$$

The existence of the optimal function $\nu^0(dv \mid t, x, u)$ solving Problem 5.2 follows from Theorem 2.1. In fact, as such a function $\nu^0(dv \mid t, x, u)$ it is sufficient to select the function-measure $\nu^\circ(dv \mid t, x)$ concentrated at the points $v = v^\circ(t, x)$, where $v^\circ(t, x)$ is the function corresponding to the optimal strategy V° solving Poblem

2 with $\varphi = \varphi_\vartheta$. The question of the existence of the function $\mu^\circ(du \mid t, x, v)$ solving Problem 5.1 cannot be answered in so simple manner. This question can once again be answered formally on the basis of Theorem 2.1 if the hypothesis in Problem 5.1 is modified somewhat. Namely, each motion $x(t, t_*, x_*, u(\cdot))$ can be broken off at the instant $\tau_{x(\cdot)} \in [t_0, \vartheta]$ at which this motion first furnishes the minimum of the quantity $\omega(t, x(t, t_0, x_0, u(\cdot))$, and next, instead of (5.5), the function $\mu^\circ(\cdot)$ is sought from the condition of the minimum of the quantity

$$\min_{\mu(\cdot)} \max_{x(\cdot)} \omega(\tau_{x(\cdot)}, x(\tau_{x(\cdot)}, t_0, x_0, \mu(\cdot))) = \varepsilon^{**}(t_0, x_0) \tag{5.7}$$

Then as the function $\mu^\circ(du \mid t, x, v)$ solving such a modified Problem 5.1 it again is sufficient to select the function-measure $\mu^\circ(du \mid t, x)$ concentrated at the points $u = u^\circ(t, x)$, where $u^\circ(t, x)$ is the function corresponding to the optimal strategy U° solving Problem 1 with $\varphi = \varphi_\vartheta$. From the fact that the strategies $U^\circ \div u^\circ(t, x)$ and $V^\circ \div v^\circ(t, x)$ form a saddle point of the game set together from Problems 1 and 2 there follows the equality $\varepsilon_*(t_0, x_0) = \varepsilon^{**}(t_0, x_0)$. At the same time the bridges W_u^ϑ and W_v^ϑ constructed on the manifolds $X_{u^\circ}^*$ and $X_{v^\circ}^*$, where in $X_{u^\circ}^*$ the motions $x(t, t_0, x_0, \mu^\circ(\cdot))$ are terminated each at its own instant $t = \tau_{x(\cdot)}$, are stable. Therefore, the solutions of problems (5.6) and (5.7) furnish optimal stable bridges W_u^ϑ and W_v^ϑ for Problems 2 and 1, respectively. However, it is understood that by such means we do not obtain a new effective method for solving the problem because under condition (1.7) problems (5.6) and (5.7) are as a matter of fact simply reformulations of Problems 2 and 1. Nevertheless, conditions (5.6) and (5.7) contain in explicit form one circumstance facilitating the solution, because, as a consequence of Theorem 2.1, to solve Problems 1 and 2 in the class of functions $u(t, x)$ and $v(t, x)$ it turns out to be sufficient to find the solutions of the analogous problems (5.7) and (5.6) in the wider class of functions $\mu(du \mid t, x, v)$ and $\nu(dv \mid t, x, u)$. To look for the optimal functions $\mu^\circ(du \mid t, x, v)$ and $\nu^\circ(dv \mid t, x, u)$ and for the bridges W_u^ϑ and W_v^ϑ generated by them, constructed on the appropriate integral manifolds $X_{u^\circ}^*$ and $X_{v^\circ}^*$, we can now use various artificial devices. One such device known is discussed in Sect. 6 ; moreover, in the regular case to be considered therein we can make use of the coarsened Problem 5.1, not replacing condition (5.5) by the more complicated condition (5.7).

Thus, in every case, the sets W_u^ϑ from (5.2) and W_v^ϑ from (5.4) are stable under condition (1.7). Therefore, the strategy $U^* \div u^*(t, x)$, extremal to the bridge W_u^ϑ from (5.2) with $\mu(\cdot) = \mu^\circ(\cdot)$ in (5.2), guarantees in the position encounter game Problem 1 a value of the functional

$$\varphi_\vartheta(x[\cdot]) \leqslant \varepsilon^*(t_0, x_0) \tag{5.8}$$

while the strategy $V^* \div v^*(t, x)$, extremal to the bridge W_v^ϑ from (5.4) with $\nu(\cdot) = \nu^\circ(\cdot)$ in (5.4), guarantees in the position evasion game Problem 2 a value of the functional

$$\varphi_\vartheta(x[\cdot]) \geqslant \varepsilon_*(t_0, x_0) \tag{5.9}$$

6. Let us discuss the construction of the *a priori* stable bridges W_u^ϑ of (5.2) and W_v^ϑ of (5.4), constructed on the integral manifolds X_u^* and X_v^* from Sect. 5, by means of imbedding these integral manifolds in certain suitable integral manifolds

X_u and X_v generated by certain contingent differential equations (see [2]). In the form to be discussed here the construction of such contingent equations was studied in [6, 7]. In the case of a linear equation of motion such constructions can be traced back to the direct method from [8,9].

Let the sets

$$H(t, x) = \bigcap_{v \in Q} \mathrm{co}\,\{f(t, x, u, v),\ u \in P\} \tag{6.1}$$

be nonempty in some region G of space $\{t, x\}$. A vector h is contained in $H(t, x)$ if and only if the inequality

$$s'h \geqslant \max_{v \in Q} \min_{u \in P} s'f(t, x, u, v) \tag{6.2}$$

is fulfilled for every choice of the vector s. We consider the function

$$\varkappa(s, t, x) = \max_{v \in Q} \min_{u \in P} s'f(t, x, u, v) \tag{6.3}$$

whose properties determine the regularity of the game. We set up an integral manifold X, set together from all solutions $x(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \vartheta)$ of the contingent differential equation

$$\dot{x} \in H(t, x) \tag{6.4}$$

assuming that this whole integral manifold is located in the region G wherein the sets $H(t, x)$ are nonempty.

Let

$$\varepsilon^\circ(t_0, x_0) = \min_{x(\cdot)} \min_{t \in T(t_0, \vartheta)} \omega(t, x(t, t_0, x_0)) = \omega(\tau^\bullet, x^\circ(\tau^\circ, t_0, x_0)),\ x(\cdot) \in X \tag{6.5}$$

We choose the solution $x^0(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ from condition (6.5). This solution forms the u-stable bridge

$$W_u^{\tau^\bullet} = [\{t, x\} : t_0 \leqslant t \leqslant \tau^\circ,\ x = x^\bullet(t, t_0, x_0)] \tag{6.6}$$

We say that the game set together from Problems 1 and 2 is regular (II) if condition (1.7) is fulfilled and if the function $\varkappa(s, t, x)$ of (6.3) is concave in s for every position $\{t, x\}$ which can be encountered on the motions $x(t, t_0, x_0)$ with the initial data $\{t_0, x_0\}$ from the region $\varepsilon^\circ(t_0, x_0) \in [\alpha, \beta]$, $t_0 \leqslant \vartheta$. (Here it is assumed that for every position $\{t_0, x_0\}$ from the selected region the quantity $\varepsilon^\circ(t_0, x_0)$ (6.5) has a meaning, i.e. the corresponding motions $x(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \vartheta)$ remain in the region G wherein the sets $H(t, x)$ are nonempty.) If the game is regular (II), the set

$$W_v^\vartheta = [\{t, x\} : t_0 \leqslant t \leqslant \vartheta,\ x = x(t, t_0, x_0) \in X] \tag{6.7}$$

forms a v-stable bridge. Thus, if the game is regular (II), then for $\varepsilon^\circ(t_0, x_0) \in [\alpha, \beta]$ the strategy $U^\circ \div u^0(t, x)$, extremal to the bridge $W_u^{\tau^\circ}$ from (6.6), guarantees in the position encounter game Problem 1 a value of the functional

$$\varphi_\vartheta(x[\cdot]) \leqslant \varepsilon^\circ(t_0, x_0) \tag{6.8}$$

while the strategy $V^\circ \div v^\circ(t, x)$, extremal to the bridge W_v^ϑ (6.7), guarantees in the position evasion game Problem 2 a value of the functional

$$\varphi_\vartheta(x[\cdot]) \geqslant \varepsilon^\circ(t_0, x_0) \tag{6.9}$$

and so, the pair of these strategies again forms a saddle point of the position game under consideration, set together from Problems 1 and 2 under the choice $\varphi = \varphi_\vartheta$.

7. The basic conclusions are formulated as follows. Suppose that the value of ϑ has been fixed. Let us assume that in region

$$\varepsilon_0 (t, x) \in (\alpha, \beta), \ t \leqslant \vartheta \tag{7.1}$$

the inequality

$$\varepsilon^\circ (t, x) \leqslant \varepsilon_0 (t, x) \tag{7.2}$$

has been fulfilled. We select a position $\{t_0, x_0\}$ from region (7.1). We fix the minimizing value τ° from condition (6.5). Let $x^\circ (t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ be the corresponding minimizing motion form (6.5). Then this solution $x^\circ (t, t_0, x_0)$ of Eq. (6.4), satisfying, therefore, the condition

$$\omega (\tau^\circ, x^\circ (\tau^\circ, t_0, x_0)) = \varepsilon^\circ (t_0, x_0) \quad (\tau^\circ \leqslant \vartheta) \tag{7.3}$$

is simultaneously the optimal program motion $x (t, t_0, x_0, \eta_\cdot^\circ)$ for Problem 4.1, corresponding to the same minimizing instant $\tau_0 = \tau^\circ$.

This statement is a consequence of the fact that as an effect of the u-stability of the bridge $W_u^{\tau^\circ}$ (6.6), constructed on the chosen solution $x^\circ (t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$, for any choice of measure $\nu_t^* (dv)$ we can find the measure $\eta_t^* (du, dv)$ connected with $\nu_t^* (dv)$ by condition (4.1) and such that the program motion $x (t, t_0, x_0, \eta_\cdot^*)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ generated by it, lies on the u-stable bridge $W_u^{\tau^\circ}$ Therefore, this program motion $x (t, t_0, x_0, \eta_\cdot^*)$ coincides with $x (t, t_0, x_0)$ for $t_0 \leqslant t \leqslant \tau^\circ$ and yields the quantity

$$\omega (\tau^\circ, x (\tau^\circ, t_0, x_0, \eta_\cdot^*)) = \omega (\tau^\circ, x^\circ (\tau^\circ, t_0, x_0)) = \varepsilon^\circ (t_0, x_0).$$

But then $\varepsilon_0(t_0 \ x_0) \leqslant \varepsilon^\circ (t_0, x_0)$. Together with the contradictory inequality (7.2) this signifies the fulfillment of the equality

$$\varepsilon_0 (t, x) = \varepsilon^\circ (t, x) \tag{7.4}$$

Since under assumption (7.2) every minimizing motion $x^\circ (t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ from problem (6.5) turns out to be simultaneously the minimizing motion $x (t, t_0, x_0, \eta_\cdot^\circ)$ from Problem 4.1, corresponding to the same value $\tau_0 = \tau^\circ$, for this motion $x (t, t_0, x_0, \eta_\cdot^\circ)$ we can compute the vector $s (t_0)$ of (4.4) wherein it is necessary to replace $\{t_*, x_*\}$ by $\{t_0, x_0\}$. In connection with this circumstance we introduce the following condition.

C o n d i t i o n 7.1. We say that this condition has been fulfilled if for every position $\{t_0, x_0\}$ *from region* (7.1), (7.2) *we can find at least one minimizing value* τ° *for problem* (6.5) such that for this value $\tau^\circ = \tau_0$ the requirement of Condition 4.1 is fulfilled. Furthermore, for this τ_0 all those minimizing program motions $x (t, t_0, x_0, \eta_\cdot^\circ)$ which coincide with one and the same motion $x^\circ (t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau_0)$ should yield one and the same value of the vector $s (t_0)$ (4.4).

We now assume that Condition 7.1 has been fulfilled. Again we select a position $\{t_0, x_0\}$ from region (7.1), (7.2) and we fix the value $\tau^\circ = \tau_0$ satisfying the requirements of Condition 7.1. Then, to this value τ^0 there corresponds only a single solution

$x^\circ(t, t_0, x_0)$ minimizing (6.5).

Indeed, let us assume the contrary. Let $x^{\circ(1)}(t, t_0, x_0)$ and $x^{\circ(2)}(t, t_0, x_0)$ be two different minimizing motions (6.4), corresponding to the slected value τ°. Further, let $\{\eta_t \mid \nu_t^\circ\}_\Pi$ be some maximizing program from (4.3), corresponding to this same value of $\tau_0 = \tau^\circ$. Again, as a consequence of the u-stability of each of the bridges $W_u^{\tau^\circ(1)}$ and $W_u^{\tau^\circ(2)}$ of form (6.6), constructed on the motions $x^{\circ(1)}(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ and $x^{\circ(2)}(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$, respectively, we conclude that in the program $\{\eta_t \mid \nu_t^\circ\}_\Pi$ we can find measures $\eta_t^{(1)}$ and $\eta_t^{(2)}$ $(t_0 \leqslant t \leqslant \tau_0 = \tau^\circ)$ such that the identities $x^{\circ(1)}(t, t_0, x_0) \equiv x(t, t_0, x_0, \eta_.^{(1)})$ and $x^{\circ(2)}(t, t_0, x_0) \equiv x(t, t_0, x_0, \eta_.^{(2)})$ are fulfilled for $t_0 \leqslant t \leqslant \tau_0 = \tau^\circ$. But this means that when $\tau_0 \in T^\circ(t_0, \vartheta)$ the maximizing program $\{\eta_t \mid \nu_t^\circ\}_\Pi$ contains two different, essentially minimizing program controls $\eta_t^{(1)}$ and $\eta_t^{(2)}$. But this contradicts Condition 4.1. The contradiction obtained proves the uniqueness of the minimizing solution $x^\circ(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$. However, then all the minimizing motions $x(t, t_0, x_0, \eta_.^\circ)$ $(t_0 \leqslant t \leqslant \tau_0 = \tau^\circ)$ of Problem 4.1 for the selected value of τ_0 coincide with one and the same motion $x(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$; therefore, it follows from Condition 7.1 that for this value of $\tau_0 = \tau^\circ$ all the minimizing motions $x(t, t_0, x_0, \eta_.^\circ)$ yield one and the same value of vector $s(t_0)$ (4.4). But in such case condition (4.6) is now fulfilled automatically for the indicated choice of $\{t_0, x_0\}$ and of $\tau^\circ = \tau_0$.

Further, in condition (6.2) let the function $\varkappa$ (6.3) be concave in s. Then, for every choice of the vector $u \in P$ the set

$$F_u(t, x, u) = \mathrm{co}\,\{f(t, x, u, v),\ v \in Q\} \tag{7.5}$$

intersects $H(t, x)$. But, according to (6.2), this signifies the fulfillment of condition (4.7). Then for every choice of $u \in P$ in set (7.5) we can find even the vector $f^* = h \in H(t, x)$ which satisfies condition (4.6) for every choice of s and not only a choice from some subset belonging to the space $\{s\}$.

Summarizing the whole discussion we arrive at the following conclusion. Suppose that inequality (7.2) is fulfilled for all positions $\{t, x\}$ from region (7.1). Then, equality (7.4) is fulfilled for positions $\{t_0, x_0\}$ from region (7.1). If here Condition 7.1 is fulfilled, then regularity (I) of the encounter-evasion game set together from Problems 1 and 2 with $\varphi = \varphi_\vartheta$ follows from regularity (II) of this game.

If the equation of motion (1.1) has the form of a properly linear Eq. (4.5) and (1.7) holds, and, moreover, if the sets $M(t)$ are convex, the additional conditions (Condition 4.1, condition (7.2) and Condition 7.1) are automatically fulfilled and, therefore, it turns out that every time the encounter-evasion game is regular (II), it is regular (I) and equality (7.4) is valid. Indeed, the fulfillment of Condition 4.1 (in the weakened form corresponding to Eq. (4.5), to condition (1.7), and to convex $M(t)$) follows directly from the convexity of $M(t)$ and of the region of attainability

$$G = [\{\tau_0, x\} : x = x(\tau_0, t_0, x_0, \eta_.), \quad \eta_. \in \{\eta_t \mid \nu_t^\circ\}_\Pi]$$

for every choice of program $\{\eta_t \mid \nu_t^\circ\}_\Pi$. Equality (7.4) follows directly from the expressions for $\varepsilon_0(t_*, x_*)$ and $\varepsilon^\circ(t_*, x_*)$ which are obtained in a well-known manner (for example, see [10])

$$\varepsilon_0(t_*, x_*) = \min_\tau \max_{\|l\|=1} \Big(\int_{t_*}^{\tau} \max_{v \in Q} \min_{u \in P} l' X(\tau, t) f(t, u, v)\, dt + l' X(\tau, t_*) x_* - \rho_{M(\tau)}(l) \Big) \quad \text{for } \varepsilon_0(t_*, x_*) \geqslant 0 \tag{7.6}$$

$$\varepsilon^\circ(t_*, x_*) = \min_\tau \max_{\|l\|=1} \left(\int_{t_*}^{\tau} \min_{h^* \in H} l'X(\tau, t) h^*(t)\, dt + l'X(\tau, t_*) x_* - \rho_{M(\tau)}(l) \right) \quad \text{for } \varepsilon^\circ(t_*, x_*) \geqslant 0 \tag{7.7}$$

Here $X(t, t_*)$ is the fundamental matrix of solutions of the homogeneous equation $\dot{x} = A(t)x$, $\rho_{M(\tau)}(l)$ is the support function of the set $M(\tau)$. $H^*(t)$ is a set of vectors

$$H^*(t) = [h^* : s'h^* \geqslant \max_{v \in Q} \min_{u \in P} s'f(t, u, v) = \varkappa^*(s, t),\ s \in \{s\}] \tag{7.8}$$

The function $\varkappa^*(s, t)$ is concave by assumption. But then, for every choice of vector s we can find a vector $h^*(s)$ satisfying the equality [11]

$$s'h^*(s) = \max_{v \in Q} \min_{u \in P} s'f(t, u, v) \tag{7.9}$$

from which follows the coincidence of the values of $\varepsilon_0(t_*, x_*)$ (7.6) and of $\varepsilon^\circ(t_*, x_*)$ (7.7). Finally, the fulfillment of Condition 7.1 in the properly linear case being considered follows from noting that in the case of Eq. (4.5) the value of vector $s(t_*)$ of (4.4) does not depend upon the choice of control η_t° if all the controls η_t° being considered, generate one and the same minimizing motion $x^\circ(t, t_0, x_0) = x(t, t_0, x_0, \eta_\cdot^\circ)$, yielding, therefore, one and the same boundary condition (4.4) for the vector-valued function $s(t)$.

In the general nonlinear case we have not succeeded in finding the condition guaranteeing the fulfillment of the additional assumptions (Conditions 4.1 and 7.1 and inequality (7.2)). However, we can point out conditions of a special form. One of these conditions is as follows. When Condition 4.1 is fulfilled, for every choice of position $\{t, x\}$ and of a nonzero vector s we can find only a single measure $\eta_t^*(du, dv)$, satisfying the condition

$$\int_P \int_Q s'f(t, x, u, v)\, \eta_t^*(du, dv) = \min_{u \in P} \int_Q s'f(t, x, u, v)\, \nu_t^*(dv) = \max_{v \in Q} \min_{u \in P} s'f(t, x, u, v) \tag{7.10}$$

the measures η_t^* and ν_t^* are connected by condition (4.1). Or we can desist from the a priori requirement of Condition 4.1, but, in this case, we must stipulate the existence of at least one minimizing instant τ° from (6.5), to which there corresponds only a single minimizing motion $x^\circ(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \tau^\circ)$ and, besides, the value m_0 from (1.4) with $t = \tau^\circ$ and $x = x^\circ(\tau^\circ, t_0, x_0)$ also must be unique. To this condition it is again sufficient to add on further the condition of uniqueness of the measure η_t^* from (7.10), the maximin condition, and that Eq. (4.4) in s satisfies the Lipschitz condition.

Finally, in the general case the relations between regularity (I) and regularity (II) of the game, in such aspects, are characterized somewhat differently. Let us assume that the game is regular (II) and that Conditiob 4.1 and condition (4.6) are fulfilled at the same time. Then $\varepsilon_0(t, x) = \varepsilon^\circ(t, x)$ for $\varepsilon_0(t, x) \in [\alpha, \beta)$ and the fulfillemt of condition (4.7) follows from regularity (II), i.e. again the game is regular (I).

Let us now return to the constructions from Sect. 5. Let the game in the general nonlinear case be regular (II). Then $\varepsilon_*(t, x) = \varepsilon^*(t, x) = \varepsilon^\circ(t, x)$ and Problem 5.1 is solved by the function $\mu^\circ(du) = \mu^\circ(du \mid t, x, v)$ which can be chosen from the condition

$$\int_P f(t, x, u, v)\,\mu^\circ(du) = \dot{x}^\circ(t, t_0, x_0) \tag{7.11}$$

for $\{t, x\} = \{t, x^\circ(t)\}$ and

$$[x - x^\circ(t)]' \int_P f(t, x, u, v)\,\mu^\circ(du) = \min_{\mu(\cdot)} [x - x^\circ(t)]' \int_P f(t, x, u, v)\,\mu(du) \tag{7.12}$$

for other $\{t, x\}$. Here $x^\circ(t) = x^*(t, t_0, x_0)$ is any one of the solutions of problem (6.5). Problem 5.2 is solved by the function $\nu^*(dv) = \nu^\circ(dv \mid t, x, u)$ which is chosen from the condition

$$\int_Q f(t, x, u, v)\,\nu^\circ(dv) \in H(t, x) \tag{7.13}$$

The validity of the assertions made follows from the remarks given below. The function $\mu^\circ(du)$, found from conditions (7.11),(7.12) which are analogous to the conditions determining the strategy extremal to the bridge $W_u^{\tau^\circ}$ of (6.6), as also the position extremal strategy, ensures the sliding along the u-stable bridge $W_u^{\tau^\circ}$ of (6.6) until the instant $t = \tau^\circ$ of all solutions $x^*(t, t_0, x_0, \mu^\circ(\cdot))$ being the limits of the Euler polygonal lines $x_\Delta^*(t, t_0, x_*, \mu^\circ(\cdot))$ of (5.1). Hence follows the inequality

$$\varepsilon^*(t_0, x_0) \leqslant \varepsilon^\circ(t_0, x_0) \tag{7.14}$$

Further, the possibility of choosing the measure $\nu^\circ(dv)$ from condition (7.13) for every choice of $u \in P$ follows from the fact that under the concavity condition of function $\varkappa(s, t, x)$ (6.3) the sets $F_v(t, x, u)$ of (7.5) and $H(t, x)$ have a nonempty intersection. But, further, all motions $x^*(t, t_0, x_0, \nu^\circ(\cdot))$, being the limits of the Euler polygonal lines $x_\Delta^*(t, t_0, x_0, \nu^\circ(\cdot))$ (5.3) for the indicated choice of the measure $\nu^\circ(dv \mid t, x, u) = \nu^\circ(dv)$ (7.13), are contained in the family of all motions $x(t, t_0, x_0)$ $(t_0 \leqslant t \leqslant \vartheta)$ of Eq. (6.4). Therefore, the v-stable bridge W_v^ϑ (5.4), constructed on the motions $x^*(t, t_0, x_0, \nu^\circ(\cdot))$, is contained in the v-stable bridge W_v^ϑ (6.7). Hence follows the inequality

$$\varepsilon_*(t_0, x_0) \geqslant \varepsilon^\circ(t_0, x_0) \tag{7.15}$$

Finally, from inequalities (7.14),(7.15) and from the fact that when the game is regular (II) $\varepsilon^\circ(t, x_0)$ is the game's value, follows the optimality of the functions $\mu^\circ(du \mid t, x, v)$ and $\nu^\circ(dv \mid t, x, u)$ indicated.

In the case of a properly linear equation of motion (4.5) the functions $\mu(du \mid t, x, v)$ and $\nu^\circ(dv \mid t, x, u)$ can be chosen analogously when the game is regular (II), but now independently of x, and then these functions acquire the meaning of upper programs from [5]. As a matter of fact, the function $\mu^\circ(du \mid t, v)$ can then be chosen from the condition

$$\int_P f(t, u, v)\,\mu^\circ(dv) = \dot{x}^\circ(t, t_0, x_0) - Ax^\circ(t, t_0, x_0)$$

while the function $\nu^\circ(dv \mid t, u)$ from the condition

$$\int_Q f(t, u, v)\,\nu^\circ(dv) = H^*(t)$$

where $H^*(t)$ is the set (7.8).

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ON AN ENCOUNTER GAME PROBLEM UNDER COMPOSITE CONTROLS

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We examine a nonlinear differential game of the encounter of a conflict-controlled phase point with a given set. We prove sufficient conditions for the successful termination of the game in the class of mixed strategies. These conditions are based on the extremal construction introduced in [1] and modified here to conform to the question being discussed.

1. Statement of the problem. Let the motion of a controlled system be described by the differential equation

$$x^{\cdot} = f(t, x, u, v) \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are r-dimensional control vectors of the first and second players, respectively, and constrained by the conditions